

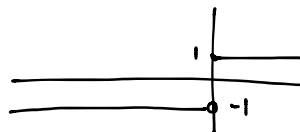
## Continuity

Def: A function  $f: X \rightarrow Y$  between topological spaces is continuous if  $f^{-1}(U)$  is open  $\forall$  open  $U \subseteq Y$ .

As we already showed, this definition agrees w/ the metric space definition.

Sanity check:

Ex: Why is  $f: \mathbb{R} \rightarrow \mathbb{R}$ , defined  $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$  discontinuous using this definition?


$$f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) = [0, \infty), \text{ not open.}$$

However,  $f$  is continuous if we give  $\mathbb{R}$  the lower limit topology!

Ex: Let  $f: \mathbb{R} \rightarrow \mathbb{R}_\ell$  be the identity function.

$f$  isn't continuous since

$f^{-1}([0, 1)) = [0, 1)$  is not open in the standard topology.  $f^{-1}$  is continuous though, since  $\mathbb{R}_\ell$  has a finer topology.

It suffices to check continuity on basis elements:

Thm:  $f: X \rightarrow Y$  is continuous iff  $f^{-1}(B)$  is open for all  $B$  in a basis for the topology on  $Y$ .

Pf: Any open set  $U \subseteq Y$  is the union of basis elements —  $U = \bigcup B_i$ , so  $f^{-1}(U) = \bigcup f^{-1}(B_i)$  is the union of open sets, hence open. The converse is immediate.  $\square$

Ex: If  $Y$  is a metric space,  $X$  any top. space, then to check  $f: X \rightarrow Y$  is continuous, it suffices to check  $f^{-1}(B_r(y))$  is open for  $y \in Y$ ,  $r > 0$ .

Ex: Let  $X$  and  $Y$  be topological spaces, and give  $X \times Y$  the product topology. Let

$$p_1: X \times Y \rightarrow X \quad \text{and} \quad p_2: X \times Y \rightarrow Y$$
$$(x, y) \mapsto x \qquad \qquad (x, y) \mapsto y$$

be the projections.

Then  $\forall U \subseteq X$  open,  $p_1^{-1}(U) = U \times Y$ , which is open, so  $p_1, p_2$  are continuous.

### Properties of continuous functions

Thm: Let  $X, Y, Z$  be top. spaces.

1.) Constant functions are continuous ( $f: X \rightarrow Y$ ,  $y_0 \in Y$ ,  $f(x) = y_0 \quad \forall x \in X$ )

- 2.) If  $A \subseteq X$ , given the subspace topology, then the inclusion  $i: A \hookrightarrow X$  is continuous.
- 3.)  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  continuous  $\Rightarrow g \circ f: X \rightarrow Z$  continuous.
- 4.) If  $X = \bigcup U_\alpha$  where  $U_\alpha$  is open, and  $f: X \rightarrow Y$  a function s.t.  $f|_{U_\alpha}$  is continuous for all  $\alpha$ , then  $f$  is continuous.

Pf of 4.) Let  $V \subseteq Y$  open. Then  $f^{-1}(V) = \bigcup f|_{U_\alpha}^{-1}(V)$

$f|_{U_\alpha}^{-1}(V)$  is open in  $U_\alpha$ , so it's the intersection of an open set in  $X$  w/  $U_\alpha$ , so it's open in  $X$ .

Thus  $f^{-1}(V)$  is the union of open sets, so it's open.  $\square$

## Homeomorphisms

Two topological spaces  $X$  and  $Y$  are "homeomorphic" if they are topologically the same.

That is, if there exists a bijection  $f: X \rightarrow Y$  s.t.

$$U \text{ is open} \iff f(U) \text{ is open.}$$

Def: A bijection  $f: X \rightarrow Y$  is a homeomorphism if  $f$  and  $f^{-1}$  are both continuous.

$X$  and  $Y$  are homeomorphic if there exists a homeomorphism

between them.

Ex: We already saw that  $f$  bijective and continuous does not imply homeomorphism:

$f: \mathbb{R}_\ell \rightarrow \mathbb{R}$ , the identity on underlying sets,

is continuous ( $(a, b) \subseteq \mathbb{R}_\ell$  is open), but  $f^{-1}$  is not continuous ( $[a, b) \subseteq \mathbb{R}$  is not open).

Ex:  $X = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$  given the topology induced by the metric on  $\mathbb{R}$ , and  $\mathbb{N} = \{0, 1, 2, \dots\}$  given discrete topology.

Define  $f: \mathbb{N} \rightarrow X$  by  $f(0) = 0, f(n) = \frac{1}{n}$  for  $n > 0$ .

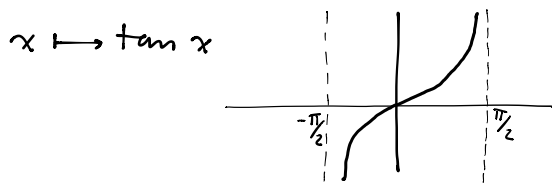
$f$  is continuous (in fact any function from a space w/ discrete top. is)

and bijective, but not a homeomorphism ( $\{0\} \subseteq X$  is not open)

On the other hand,  $f: \{1, 2, \dots\} \rightarrow \{1, \frac{1}{2}, \dots\}$ , defined  $f(n) = \frac{1}{n}$  is a homeomorphism — we can find an open ball around  $\frac{1}{n}$  not containing  $\frac{1}{n+1}$  or  $\frac{1}{n-1}$ , so  $\{1, \frac{1}{2}, \dots\}$  has discrete topology.

A metric space is bounded if  $\sup\{d(x, y)\} < \infty$ . We can't detect this topologically:

Ex: Consider the bijection  $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$  defined



$f$  and  $f^{-1}$  are continuous, so  $(-\frac{\pi}{2}, \frac{\pi}{2})$  is homeomorphic to  $\mathbb{R}$ .

In fact, since all open intervals are homeomorphic via linear maps,  $\mathbb{R}$  is homeomorphic to every open interval.

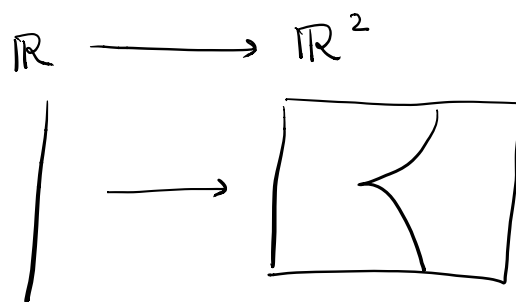
Def: If  $f: Y \rightarrow X$  is an injective continuous map, then  $f$  is an embedding if the map  $Y \rightarrow f(Y)$  is a homeomorphism (w/  $f(Y) \subseteq X$  given the subspace topology)

Ex:  $f: \mathbb{N} \rightarrow \mathbb{R}$  defined

$$f(x) = \begin{cases} 0 & \text{for } x=0 \\ 1/x & \text{otherwise} \end{cases} \quad \text{is not an embedding}$$

since the image doesn't have the discrete topology.

Caution: Embedding has a different meaning in differential topology...



This is a topological embedding but not a "smooth embedding"